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Notes on Three Integral Dependence Theorems*

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If P is a prime ideal in an integral extension domain A of a quasi-local domain R and if F is the quotient field of R, then the following results hold when R is integrally closed: height $P =$ height $P \cap R$; altitude $A_P =$ altitude $A_P \cap F$; $(A_P)' \cap F = (A_P \cap F)'$, where the prime denotes integral closure. This paper is concerned with generalizing these results to the case where R is not integrally closed.

1. INTRODUCTION

All rings in this paper are assumed to be commutative with identity, and the terminology is, in general, the same as that in [6]. A' will always be used to denote the integral closure of a ring A in its total quotient ring.

Throughout this section, let R be a quasi-local domain with quotient field F, let A be an integral extension domain of R, and let N be a maximal ideal in A. In [6, (10.14)], it is shown that if R is integrally closed, then height $N =$ height $N \cap R$. On the other hand, [6, Ex. 2, pp. 203–205] shows that this need not hold when R is not integrally closed. Since this is a useful property, in the hope of obtaining a generalization, it was asked in [9, (2.3.1)] if it is always true that height $N =$ height $N \cap (A \cap R')$. The first theorem in this paper, (2.1), shows that the answer is, in general, no—even when R is Noetherian and $A = R[b]$ is a simple integral extension domain of R. After proving some corollaries that give several characterizations of when this does hold for a given Noetherian domain, Section 2 is closed with an example, (2.6), that shows an unexpected hypothesis in (2.1) is necessary.

In Section 3, it is conjectured that altitude $A_N \cap F =$ altitude A_N, and two important cases when this holds are given: R is Noetherian and all maximal ideals in R' have the same height (3.1) (this holds for all local domains of classical algebraic geometry); and, for each maximal ideal M' in R' there

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exists a maximal ideal N' in A' such that $N' \cap A = N$ and $N' \cap R' = M'$ (3.3).

In Section 4, the stronger conjecture that $(A_N)' \cap F$ is integral over $A_N \cap F$ is considered. Among other results, it is shown that this holds if A has only finitely many maximal ideals when R is Noetherian and height $N = 1$ (4.16), and it also holds when A and height N are arbitrary and R satisfies condition (QSM) (4.12.3) (see (4.10) and (4.24)). (If R' has only finitely many maximal ideals, then there always exist quasi-local integral extension domains $R^* \subseteq R'$ of R such that R^* is contained in a finite R-module and R^* satisfies condition (QSM) (4.11.4). Each of these cases implies the following interesting result: a necessary condition for A_N to be R-flat is that the condition in (3.3) (mentioned above) must be satisfied (4.13). Section 4 is closed by showing, in (4.18)–(4.24), that a number of additional statements either imply or are equivalent to $(A_N)' \cap F = (A_N \cap F)'$.

The results in Sections 3 and 4 fall considerably short of verifying the conjectures considered. But the conjectures are verified for a number of important special cases, and a number of additional useful results are proved, so it is hoped that the material in these sections will be of sufficient interest and importance to merit consideration by others.

2. Notes on height $P =$ height $P \cap (B \cap A')$

In this section, we consider the following question which was asked in [9, (2.3.1), 11, (15.4.1)(a)]:

(CIC) *If B is an integral extension domain of an integral domain A and if P is a prime ideal in B, then must* height $P =$ height $P \cap (B \cap A')$?

(Equivalently, is height $P =$ height $P \cap (B \cap F)$ where F is the quotient field of A?) The answer is readily seen to be yes when A is a nice ring—for example, when A satisfies the altitude formula, but (2.1) shows that in general the answer to (CIC) is no—even when A is local (Noetherian) and B is a simple integral extension domain of A. (At first glance it seems hypothesis (i) in (2.1) may be superfluous, but it is, in fact, necessary, as is shown in (2.6). Also, some equivalences of the assumption height $p' <$ height $p' \cap A$ are given in (2.3).)

(2.1) Proposition. *Let A be an integral domain and let p' be a prime ideal in A' such that*: (i) *$A'_{p'}$ is not Henselian and* (ii) *$p' \not\supseteq \cap \{P' \in \operatorname{Spec} A'$; $P' \cap A = p' \cap A$ and $P' \neq p'\}$. Assume* height $p' <$ height $p' \cap A$. *Then there exists a simple integral extension domain $B = A[b]$ of A that has a prime ideal Q such that* height $Q =$ height p', $Q \cap A = p' \cap A$ (*so* depth $Q =$ depth p'), *and* height $Q <$ height $Q \cap (B \cap A')$.

Proof. By (i), there exists a finite integral extension domain C of A' that has at least two prime ideals that lie over p'. Let $Q_1,..., Q_n$ be all these prime ideals in C. Then, by (ii), $Q_1 \not\supseteq I = \bigcap \{Q \in \operatorname{Spec} C;\ Q \cap A = p' \cap A$ and $Q \notin \{Q_1,..., Q_n\}\}$, so $Q_1 \not\supseteq I \cap Q_2 \cap \cdots \cap Q_n$, since Q_1 is prime. Therefore there exists an element $b \in I \cap Q_2 \cap \cdots \cap Q_n$ such that $b \notin Q_1$. Let $B = A[b]$ and let $Q = Q_1 \cap B$.

Then B has exactly two prime ideals that lie over $p' \cap A$, namely, Q and $Q_2 \cap B$. Also, height Q = height Q_1, since Q_1 is the only prime ideal in C that lies over Q, and height Q_1 = height p', by [6, (10.14)], so height Q = height p'. Moreover, $Q \cap A = Q_1 \cap A = p' \cap A$, so depth Q = depth p' and height $Q <$ height $Q \cap A$ (by the assumption on p'). Let $D = B \cap A'$. Then it remains to show that height $Q <$ height $Q \cap D$. For this, it will be shown that $Q \cap D = (Q_2 \cap B) \cap D$, so $Q \cap D$ is the only prime ideal in D that lies over $p' \cap A$, hence height $Q \cap D$ = height $p' \cap A$ = height $Q \cap A >$ height Q.

To show that $Q \cap D = Q_2 \cap D$, let $x \in Q \cap D$. Then x is in Q and in A', so $x \in Q_1 \cap A' = Q_2 \cap A'$. Therefore $x \in Q_2$, so $x \in Q_2 \cap D$. Thus $Q_1 \cap D \subseteq Q_2 \cap D$ and $(Q_1 \cap D) \cap A = p' \cap A = (Q_2 \cap D) \cap A$, so it follows from integral dependence that $Q_1 \cap D = Q_2 \cap D$. Q.E.D.

(2.2) *Remarks.* (2.2.1) It is clear from (2.1) that B is a simple integral extension domain of $B \cap A'$ and B is not a flat $B \cap A'$-algebra. (On the other hand, a simple integral extension domain of A' is free.)

(2.2.2) The proof of (2.1) shows that if A is local and $p' \cap A$ is its maximal ideal, then $B \cap A'$ is a quasi-local integral extension domain of A. Therefore if there are no rings properly between A and A' and if A' has two maximal ideals, then $B \cap A' = A$. This holds, for example, when A is as in [6, Ex. 2, pp. 203–205].

For a prime ideal p in a Noetherian domain A, (2.3) gives several equivalences for the existence of a prime ideal P in some integral extension domain of A to satisfy $P \cap A = p$ and height $P <$ height $P \cap (B \cap A')$. Actually, for all Noetherian domains it is quite well known that (2.3.2)–(2.3.4) are equivalent and each is implied by (2.3.1), so it is only in showing that (2.3.3) $\Rightarrow$ (2.3.1) that the Henselian hypothesis is used. The main reason for including all four statements in (2.3) is for ease of reference, since it is a useful result in applications. Note that since A is Noetherian in (2.3), hypothesis (ii) in (2.1) is satisfied.

(2.3) THEOREM. *Let p be a prime ideal in a Noetherian domain A and assume that, for each prime ideal p' in A' that lies over p, $A'_{p'}$ is not Henselian. Then the following statements are equivalent*:

(2.3.1) *There exists an integral extension domain B of A and a prime ideal P in B such that $P \cap A = p$ and* height $P <$ height $P \cap (B \cap A')$.

(2.3.2) *There exists an integral extension domain B of A and a prime ideal P in B such that $P \cap A = p$ and* height $P <$ *height* p.

(2.3.3) *There exists a prime ideal p' in A' such that $p' \cap A = p$ and* height $p' <$ height p.

(2.3.4) *There exists a simple integral extension domain $A[b] \subseteq A'$ of A and a prime ideal q in $A[b]$ such that $q \cap A = p$ and* height $q <$ height p.

Proof. Assume (2.3.4) holds and let $p' \in \operatorname{Spec} A'$ such that $p' \cap A[b] = q$. Then height $p' \leqslant$ height q, so this p' satisfies the conclusion of (2.3.3).

(2.3.3) $\Rightarrow$ (2.3.1), by (2.1), and (2.3.1) $\Rightarrow$ (2.3.2), since height $P \cap (B \cap A') \leqslant \text{height}(P \cap (B \cap A')) \cap A$.

Assume (2.3.2) holds and let P' be a prime ideal in B' such that $P' \cap B = P$ and height $P' =$ height P. Let $p' = P' \cap A'$. Then height $p' =$ height P', by [6, (10.14)], and $p' \cap A = p$, so it follows that (2.3.2) $\Rightarrow$ (2.3.3).

Finally, assume (2.3.3) holds and let $b \in p'$ such that b is not in any other prime ideal in A' that lies over p. Then height $p' =$ height $p' \cap A[b]$, so b and $q = p' \cap A[b]$ satisfy the conclusions of (2.3.4). Q.E.D.

(2.4), which follows immediately from (2.3), gives several necessary and sufficient conditions for (CIC) to have an affirmative answer for a given Noetherian domain A.

(2.4) COROLLARY. *Let A be a Noetherian domain such that for each nonzero prime ideal p' in A', $A'_{p'}$ is not Henselian. Then the following statements are equivalent*:

(2.4.1) *For every integral extension domain B of A and for every prime ideal P in B*, height $P =$ height $P \cap (B \cap A')$.

(2.4.2) *For every integral extension domain B of A and for every prime ideal P in B*, height $P =$ height $P \cap A$.

(2.4.3) *For every prime ideal p' in A'*, height $p' =$ height $p' \cap A$.

(2.4.4) *For every $b \in A'$ and for every prime ideal q in $A[b]$*, height $q =$ height $q \cap A$.

Hypothesis (2.1) (ii) is not needed in (2.4) since A is Noetherian. The next corollary gives a case when neither (i) nor (ii) in (2.1) is needed.

(2.5) COROLLARY. *The following statements are equivalent for a Noetherian domain A*:

(2.5.1) *For every integral extension domain C of $A[X]$ and for every prime ideal Q in C*, height $Q =$ height $Q \cap (C \cap A[X]')$.

(2.5.2) *For every integral extension domain C of $A[X]$ and for every prime ideal Q in C*, height $Q =$ height $Q \cap A[X]$.

(2.5.3) *For every prime ideal P' in $A[X]'$*, height $P' =$ height $P' \cap A[X]$.

(2.5.4) *For every $f \in A[X]'$ and for every prime ideal q' in $A[X, f]$*, height $q' =$ height $q' \cap A[X]$.

(2.5.5) *For every prime ideal p' in A'*, height $p' =$ height $p' \cap A$.

Proof. The equivalence of (2.5.1)–(2.5.4) is clear by (2.4), since $(A[X]')_{P'}$ is not Henselian when $P' \neq (0)$.

If (2.5.5) holds, then let P' be a prime ideal in $A[X]'$, let $p' = P' \cap A'$, and let $p = p' \cap A$. If $P' = p'A[X]'$, then height $P' =$ height $p' =$ height $p =$ height $pA[X]$, by [10, (2.2)] and hypothesis, and $P' \cap A[X] = pA[X]$, so height $P' =$ height $P' \cap A[X]$. If $P' \supset p'A[X]'$, then height $P' \geqslant$ height $p' + 1$ and $P = P' \cap A[X] \supset pA[X]$. Therefore, since $P \cap A = p$, height $P =$ height $p + 1 =$ height $p' + 1 \leqslant$ height P', and height $P' \leqslant$ height P always holds, so (2.5.5) $\Rightarrow$ (2.5.3). The converse follows by a similar (and easier) argument. Q.E.D.

This section will be closed by showing that hypothesis (i) in (2.1) is necessary. (The proof of (2.6) uses a result that will be proved in Section 4 of this paper.)

(2.6) EXAMPLE. There exists a quasi-local domain R such that there exists a maximal ideal M in R' such that R'_M is Henselian and such that for all integral extension domains A of R that have only finitely many maximal ideals and for all maximal ideals Q in A such that height $Q <$ altitude R, height $Q =$ height $Q \cap (A \cap R')$. Namely, let (L, p) be as in [6, Ex. 2, pp. 203–205] in the case $m \geqslant 0$ and $r \geqslant 1$, so L is a local domain of altitude $r + m + 1$, L' has exactly two maximal ideals, say P and N, and height $P = m + 1$ and height $N = r + m + 1$. Let L^* be the separable integral closure of L'_P in an algebraic closure of its quotient field, let P^* be a maximal ideal in L^*, let L'' be the splitting ring of P^*, and let $P'' = P^* \cap L''$, so $L''_{P''}$ is the Henselization of L'_P. (See [6, p. 180].) Let I be the integral closure of L in the quotient field of L'' (so $L'' = I_{(L'-P)}$), let $P' = P'' \cap I$, and let N' be a maximal ideal in I that lies over N. Let $S = I_{(I-(P' \cup N'))}$, and let $M = P'S$ and $K = N'S$. Then $S_M = L''_{P''}$ is Henselian, altitude $S_M =$ height $P = m + 1$, and altitude $S_K =$ height $N = r + m + 1$. Let $R = L + J'$, where $J' = M \cap K$ is the Jacobson radical of S. Then R is a quasi-local domain and $R' = S$ (since, with $B = L + (P' \cap N')$ and $J_0 = p + (P' \cap N')$, $B' = I$, so $S = (B_{J_0})'$, and R contains B_{J_0}). Now let A be an arbitrary integral extension domain of R that has only finitely many maximal ideals and that has a maximal ideal Q such that height $Q < r + m + 1$. Then, using [6, (10.14)] it readily follows that the maximal ideals in any integral extension domain of L have height either $m+1$ or $r+m+1$, so it follows that height $Q = m+1$. Also, every other maximal ideal in A must have height $r+m+1$, for each maximal ideal in A' of height $m+1$ must contract in $R' = S$ to M, by [6, (10.14)], and S_M is Henselian. Therefore let $x \in M$ such that $1-x \in K$. Then $x^2 - x \in M \cap K \subseteq R$, so x is quadratic over R. Let $D_1 = R[x]_{M \cap R[x]}$ and $D_2 = R[x]_{K \cap R[x]}$, so $(D_1)' = S_M$ and $(D_2)' = S_K$. Let F be the quotient field of R and let $Q_1, \dots, Q_n$ be the other maximal ideals in A. Then, by (4.9.2), $A_Q \cap F \supseteq D_1$ and each $A_{Q_i} \cap F \supseteq D_2$, so $R[x] \subseteq A \cap F$. Therefore $R[x] \subseteq A \cap R'$, so $Q \cap (A \cap R') \supseteq Q \cap R[x] = M \cap R[x]$, so $m+1 =$ height $Q \leqslant$ height $Q \cap (A \cap R') \leqslant$ height $M \cap R[x] =$ height $M = m+1$, hence height $Q =$ height $Q \cap (A \cap R')$.

3. Notes on altitude $A_N =$ altitude $A_N \cap F$

It was shown in Section 2 that the answer to (CIC) is, in general, no. In this brief section, and to some extent in the next, we consider the relationship between altitude A_P and altitude $A_P \cap F$, where A is an integral extension domain of an integral domain R, P is a prime ideal in A, and F is the quotient field of R. Actually, since $R_{P \cap R} \subseteq R_{P \cap R}[A]$ and $PA_P \cap R_{P \cap R}[A]$ satisfy the conditions on R, A, and P, we restrict attention to the case R is quasi-local and P is maximal, and we consider the following question:

(CQF) *Is* altitude $A_N =$ altitude $A_N \cap F$ *when A is an integral extension domain of a quasi-local domain R with quotient field F and N is a maximal ideal in A?*

We note first, in (3.1), that the answer is yes for all rings of classical algebraic geometry (see (3.2)).

To avoid continual repetition, *we fix the notation as in* (CQF) *and also let M be the maximal ideal in R.* Also, for (3.1), we need to use the *valuative altitude*, denoted altitude$_v R = \sup\{$altitude R_v; R_v is a valuation ring in the quotient field of R and $R \subseteq R_v\}$.

(3.1) Proposition. *Assume that* height $M' =$ altitude R *for all maximal ideals M' in R' and that* altitude$_v R =$ altitude R. *Then* altitude $A_N \cap F =$ altitude $A_N =$ altitude R.

Proof. Let N' be a maximal ideal in A' that lies over N. Then, by integral dependence, height $N \leqslant$ height $M =$ height $N' \cap R'$ (by hypothesis) height $N' \leqslant$ height N (by [6, (10.14)]), so altitude $A_N =$ altitude R. Let $L = A_N \cap F$ and let $(0) \subset P_1 \subset \cdots \subset P_n = NA_N$ be a chain of prime ideals in A_N. Then $(0) \subset P_1 \cap L \subset \cdots \subset P_n \cap L$, since A is integral over R. Therefore altitude $L \geqslant$ altitude $A_N =$ altitude R. However, since altitude$_{\iota'}R =$ altitude R, it follows from [6, (11.9)] that altitude $L \leqslant$ altitude R. Q.E.D.

(3.2) *Remark.* It seems likely that (3.1) should hold without the assumption altitude$_{\iota'}R =$ altitude R, but I have not been able to prove this. In any case, if R is integrally dependent on a Noetherian ring, then altitude$_{\iota'}R =$ altitude R, by [5, Corollary 2, p. 67 and Proposition 4, p. 58], so (3.1) holds for local domains R such that each maximal ideal in R' has the same height. In particular, this holds for all local domains that satisfy the second chain condition for prime ideals, so it holds for all local domains of classical algebraic geometry.

One further case when the answer to (CQF) is yes will be given in (3.3), and some additional cases will be given in (4.9), (4.12), (4.16), (4.19), and (4.22).

(3.3) PROPOSITION. *If for each maximal ideal M' in R' there exists a maximal ideal N' in A' such that $N' \cap A = N$ and $N' \cap R' = M'$, then $R \subseteq A_N \cap F \subseteq (A_N)' \cap F = R'$, so* altitude $A_N \cap F =$ altitude $A_N =$ altitude R.

Proof. $R \subseteq A_N \cap F \subseteq (A_N)' \cap F = \bigcap \{A'_{N'}; N'$ is a maximal ideal in A' and $N' \cap A = N\} \cap F = \bigcap \{R'_{M'}; M'$ is a maximal ideal in $R'\}$ (by [2, (12.7)]) $= R'$ (by [6, (33.8)] and hypothesis). Thus, by integral dependence, each of these rings has the same altitude. And, if M' is a maximal ideal in R' such that height $M' =$ height M and if N' is a maximal ideal in $(A_N)'$ that lies over M', then height $M' =$ height N' by [6, (10.14)], and height $N' \leqslant$ height $N \leqslant$ height M, so it follows that altitude $A_N =$ altitude R. Q.E.D.

(3.4) COROLLARY. *If R' is quasi-local, then $R \subseteq A_N \cap F \subseteq (A_N)' \cap F = R'$, so* altitude $A_N \cap F =$ altitude $A_N =$ altitude R.

Proof. This is clearly by (3.3). Q.E.D.

In closing this section, it should be noted that the hypothesis in (3.4) is satisfied when R is Henselian and when $R = R'$.

4. NOTES ON $(A_N \cap F)' = (A_N)' \cap F$

The following result, which was used in the proof of (3.3), has appeared (in various forms) in several papers and books (for example, [1, Lemma 1.29; 2, (12.7); 4, Theorem 1; 7, Lemma 1.3]): if A is an integral extension domain of an integrally closed integral domain R and if N is a maximal ideal in A, then $A_N \cap F = R_{N \cap R}$ (so it follows that $(A_N)' \cap F = R_{N \cap R} = (A_N \cap F)'$), where F is the quotient field of R. This is a very useful result, and it is natural to ask how the conclusion should be changed when it is not assumed that R is integrally closed. Using the integrally closed case, it seems clear that $(A_N \cap F)'$ should be $(A_N)' \cap F$. Then, if R' has only finitely many maximal ideals, this would immediately imply altitude $A_N =$ altitude $A_N \cap F$ (by (4.2) and integral dependence), so the answer to (CQF) is yes. However, the "clear" answer seems to be not easy to prove. Even so, there are some important cases when this answer can be shown to hold, and these cases are considered in this section. We begin by fixing some notation. (Many of the results below can be proved in a more general case, but to avoid frequent repetitions of certain hypotheses, it is probably best to generally restrict attention to the case described in (4.1).)

(4.1) *Remark.* The following notation is fixed through (4.9): (R, M) is a quasi-local domain such that there are only finitely many maximal ideals in R', F is the quotient field of R, A is an integral extension domain of R that has only finitely many maximal ideals, $\mathscr{S} = \{N_1, ..., N_h\}$ is a finite set of maximal ideals in A, $S = A - \bigcup_1^h N_j$, $\mathscr{S}' = \{N'; N'$ is a maximal ideal in A' and $N' \cap A \in \mathscr{S}\}$, $\mathscr{M} = \{N' \cap R'; N' \in \mathscr{S}\} = \{M'_1, ..., M'_g\}$, and $\mathscr{M}'$ is the set of all maximal ideals in R'.

Concerning the restriction that A has only finitely many maximal ideals, note that if J'' is the Jacobson radical of the integral closure R'' of R in a given algebraic extension field E of F, then $Q = R + J''$ is a quasi-local integral extension domain of R with quotient field E, so finite integral extension domains of Q contained in R'' satisfy the conditions on A.

Several preliminary results are needed to prove (4.9), the first of the main results in this section, and among these are the following lemma and definition.

(4.2) Lemma. *With the notation of* (4.1), $(A_S)' \cap F = \bigcap \{R'_{M'}; M' \in \mathscr{M}\}$ *and* altitude $A_S =$ altitude$(A_S)' =$ altitude$(A_S)' \cap F$.

Proof. $(A_S)' = \bigcap \{A'_{N'}; N' \in \mathscr{S}'\}$, by [6, (33.8)], and, for each $N' \in \mathscr{S}'$, $A'_{N'} \cap F = R'_{N' \cap R'}$, by [2, (12.7)]. Therefore it follows that $(A_S)' \cap F = \bigcap \{R'_{M'}; M' \in \mathscr{M}\}$. Also, altitude $A_S =$ altitude$(A_S)'$, by integral dependence, and, for each $N' \in \mathscr{S}'$, altitude $A'_{N'} =$ altitude $A'_{N'} \cap F$, by [6, (10.14)]. Therefore altitude$(A_S)' =$ altitude$(A_S)' \cap F$, since if B is a ring, then altitude $B = \sup\{$altitude B_Q; Q is a maximal ideal in $B\}$. Q.E.D.

(4.3) Definition. If $(B; M_1, ..., M_h)$ is a quasi-semi-local ring, then the *Henselization* B^H of B is defined to be the direct sum of the Henselizations $(B_{M_i})^H$ of the B_{M_i} $(i = 1, ..., h)$.

The reader will find the definition of the Henselization of a quasi-local ring in [6, p. 180] and [6, Section 43] contains many useful properties of the Henselization of a quasi-local ring which will be used below.

(4.4) contains two rather obvious statements concerning Henselizations, but certain proofs below will be clarified by having these explicit statements for reference.

(4.4) *Remark.* With the notation of (4.1), the following statements hold:

(4.4.1) $R^{H'} = R'^{H} = \oplus\{(R'_{M'})^H : M' \in \mathscr{M}'\}$ and each $(R'_{M'})^H$ is a quasi-local domain.

(4.4.2) Let $z_1,\ldots,z_k$ be the minimal prime ideals in R^H and let $M'_1,\ldots,M'_k$ be the maximal ideals in R' arranged such that $(R'_{M'_i})^H = (R^H/z_i)'$ $(i = 1,\ldots,k)$ (see [6, (43.20) and Ex. 2, p. 188]). Then, for each g $(1 \leqslant g \leqslant h)$, $(R^H/(\bigcap_1^g z_i))' = \oplus_1^g (R'_{M'_i})^H = (R'_{S_0})^H = R^{H'}/Z'$, where $S_0 = R' - \bigcup_1^g M'_i$ and $Z' = (\bigcap_1^g z_i)\, T_0 \cap R^{H'}$ with T_0 the total quotient ring of R^H.

Proof. By [6, (43.20) and Ex. 2, p. 188], $R^{H'} = \oplus\{(R'_{M'})^H;\ M'$ is a maximal ideal in $R'\}$, so (4.4.1) follows from the definitions of R^H with R quasi-semi-local and with R quasi-local and integrally closed.

(4.4.2) The total quotient ring of $R^H/(\bigcap_1^g z_i)$ is $T_0/(\bigcap_1^g z_i)T_0$, so $(R^H/(\bigcap_1^g z_i))' = \oplus_1^g (R'_{M'_i})^H = R^{H'}/((\bigcap_1^g z_i)\ T_0 \cap R^{H'})$, by (4.4.1), so (4.4.2) follows from the definition of $(R'_{S_0})^H$. Q.E.D.

(4.5) LEMMA. *With the notation of* (4.1), *let* (L, N) *be a quasi-local integral extension domain of* R. *Then* L^H *contains and is integral over* R^H *and minimal prime ideals in* L^H *lie over minimal prime ideals in* R^H.

Proof. The zero ideals in R^H and L^H are *semi-prime* (that is, intersections of prime ideals), by [6, (43.20)]. Also there exists a homomorphism τ on R_H into L^H such that L^H contains and is integral over $\tau(R^H)$, by [6, (43.17)] and its proof. Therefore Ker τ is semi-prime. Let w be a minimal prime ideal in L^H and let $z = w \cap \tau(R^H)$. Then $w \cap L = (0)$, by [6, (43.20)], so $w \cap R = (0)$, and so $\tau^{-1}(z) \cap R = (0)$. Therefore, since R^H is a localization of an integral extension ring of R, it follows that $\tau^{-1}(z)$ is a minimal prime ideal, so z is a minimal prime ideal, and so minimal prime ideals in L^H lie over minimal prime ideals in $\tau(R^H)$. On the other hand, if z is a given minimal prime ideal in $\tau(R^H)$, then by integral dependence there exists a minimal prime ideal w in L^H that lies over z, and so by what was just shown it follows that Ker τ is a (finite) intersection of minimal prime ideals in R^H. Therefore, if it is shown that $(\tau(R^H))'$ has as many maximal ideals as $R^{H'}$, then it follows from (4.4.1) and (4.4.2) that Ker $\tau = (0)$. For this, let M' be a maximal ideal in R'. Then there exists a maximal ideal N' in L' that lies over M', and then there exists a maximal ideal $N^{*\prime}$ in $L^{H'}$ that lies over N'. Therefore $N^{*\prime} \cap (\tau(R^H))'$ lies over M', so it follows that $\tau(R^H)'$ and $R^{H'}$ have the same number of maximal ideals, and so Ker $\tau = (0)$. Q.E.D.

By [6, (43.5) and (43.18)], if a quasi-local ring (L^*, N^*) *dominates* a quasi-local ring (L, N) (that is, L is a subring of L^* and $N^* \cap L = N$) and if L^* is a quotient ring of a finite integral extension ring of L, then there exists a unique homomorphism σ from L^H into L^{*H} such that L^{*H} dominates and is integral over $\sigma(L^H)$. (4.6) describes the kernel of σ in a somewhat more general case. (In (4.6) and the remainder of this paper, $\mathrm{Ker}(R^H \to (A_{N_j})^H)$ is the kernel of this homomorphism σ, and $\mathrm{Ker}(R^H \to (A_S)^H) = \bigcap_1^h (\mathrm{Ker}(R^H \to (A_{N_j})^H))$ (since $(A_S)^H = \bigoplus_1^h (A_{N_j})^H$).)

(4.6) PROPOSITION. *With the notation of* (4.1), *for* $i = 1,..., g$ *let* z_i *be the minimal prime ideal in* R^H *such that* $(R^H/z_i)' = (R'_{M'_i})^H$, *and for* $j = 1,..., h$ *let* $Z_j = \mathrm{Ker}(R^H \to (A_{N_j})^H)$. *Then* $Z_j = \bigcap \{z_i$; *there exists* $N' \in \mathscr{S}'$ *such that* $N' \cap A = N_j$ *and* $N' \cap R' = M'_i\}$ *and minimal prime ideals in* $(A_{N_j})^H$ *lie over minimal prime ideals in* R^H/Z_j. *Moreover*, $\mathrm{Ker}(R^H \to (A_S)^H) = \bigcap_1^g z_i$.

Proof. It will first be shown that it may be assumed that A is a finite integral extension domain of R. For this, let $N_1,..., N_m$ be all the maximal ideals in A and let $a_1,..., a_m$ in A such that $B = R[a_1,..., a_m]$ has m maximal ideals. Then, with $Q_j = N_j \cap B$, A_{N_j} is integral over B_{Q_j}. Let $\mathscr{E} = \{Q_1,..., Q_h\}$ and let $\mathscr{E}' = \{N' \cap B'; N' \in \mathscr{S}'\}$. Then if Q' is a maximal ideal in B', then $Q' \cap B \in \mathscr{E}$ if and only if $Q' \in \mathscr{E}'$, since A and B have the same number of maximal ideals (and by integral dependence and the definitions of $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{E}$, and $\mathscr{E}'$). Also, it readily follows that for $j = 1,..., h$ and $i = 1,..., g$ if there exists a maximal ideal N' in A' such that $N' \cap A = N_j$ and $N' \cap R = M'_i$, then $Q' = N' \cap B'$ satisfies $Q' \cap B = Q_j$ and $Q' \cap R' = M'_i$, and conversely. Therefore by (4.5) (applied to B_{Q_j} and A_{N_j} in place of R and L, respectively) and the uniqueness of $R^H \to (A_{N_j})^H$ (see [6, (43.5)]), it may be assumed that A is a finite integral extension domain of R. Therefore A' has only finitely many maximal ideals, since R' does.

Fix $i = 1,..., g$, let $M' = M'_i$, and let $N'_1,..., N'_e$ be the maximal ideals in A' that lie over M' arranged such that $N'_1,..., N'_d$ are in $\mathscr{S}'$. Let $a_1,..., a_p$ generate A over R and for $i = 1,..., e$ let $x_i \in \bigcap \{N'_j;\ j = 1,..., e$ and $j \neq i\} - N'_i$. Let $C = R'_{M'}[a_1,..., a_p, x_1,..., x_e]$. Then $A'_{(R'-M')}$ is integral over C and the ideals $Q_j = N'_j A'_{(R'-M')} \cap C$ are distinct, so $(C_{Q_j})' = A'_{N'_j}$ $(j = 1,..., e)$. Therefore, since C is finitely generated over $R'_{M'}$, each $(C_{Q_j})^H$ contains and is integral over $(R'_{M'})^H$, by [6, (43.18)] (and since $(R'_{M'})^H$ and $(C_{Q_j})^H$ are domains and altitude L = altitude L^H, when L is quasi-local), and $(C_{Q_j})^H \subseteq (C_{Q_j})^{H'} = (C_{Q_j})'^H = (A'_{N'_j})^H$, by (4.4.1). Therefore each $(A'_{N'_j})^H$ contains and is integral over $(R'_{M'})^H$. This will be used in the next paragraph.

Now fix $j = 1,..., h$ and let $W_j = \mathrm{Ker}((A_S)^H \to (A_{N_j})^H)$. Then W_j is a finite intersection of minimal prime ideals in $(A_S)^H$, since $(A_{N_j})^H$ is a direct summand of $(A_S)^H$ and $(A_S)^H$ has no nonzero nilpotent elements (by [6, (43.20)]). Also, by [6, (43.5) and (43.18)], $(A_S)^H/W_j = (A_{N_j})^H =$ (say) A_j

contains and is integral over $R^H/Z_j =$ (say) R_j, so it follows that Z_j is a finite intersection of prime ideals in R^H. In fact, it follows as in the proof of (4.5) that Z_j is a finite intersection of minimal prime ideals in R^H and minimal prime ideals in A_j lie over minimal prime ideals in R_j. Therefore, let z_i be a minimal prime ideal in R^H that contains Z_j and let M'_i be the corresponding maximal ideal in R'. Let $\bar{w}$ be a minimal prime ideal in A_j that lies over $\bar{z} = z_i/Z_j$, let N^* be the maximal ideal in $(A_{N_j})'$ that corresponds to $\bar{w}$, and let $N' = N^* \cap A'$. Then $(A'_{N'})^H = (A_j/\bar{w})'$ contains and is integral over $(R_j/\bar{z})' = (R^H/z_i)' = (R'_{M'_i})^H$, so it follows that $N' \cap A = N_j$ and $N' \cap R' = M'_i$. Therefore $Z_j \supseteq \bigcap \{z_i$; there exists $N' \in \mathscr{S}'$ such that $N' \cap A = N_j$ and $N' \cap R' = M'_i\} =$ (say) Z^*. On the other hand, if z is a minimal prime ideal in R^H that contains Z^*, then let $z = z_i$ and let N', N_j, and M'_i be the maximal ideals given by $z_i \supseteq Z^*$. Then $(A'_{N'})^H$ contains and is integral over $(R'_{M'_i})^H$, as in the preceding paragraph, and $(A'_{N'})^H$ is a direct summand of $A'_j = (A_{N_j})^{H'}$. Let w' be the minimal prime ideal in A'_j such that $A'_j/w' = (A'_{N'})^H$, let $\bar{z} = w' \cap R_j$, and let p be the preimage in R^H of $\bar{z}$. Then A'_j/w' cntains and is integral over both $R^H/p = R_j/\bar{z}$ and R^H/z_i (since $A'_j/w' = (A'_{N'})^H \supseteq (R'_{M'_i})^H = (R^H/z_i)' \supseteq R^H/z_i$). Therefore, by the uniqueness of $R^H \to A'_j/w'$ (see [6, (43.5)] and note that R is dominated by $A'_j/w' = (A'_{N'})^H$), it follows that $z = z_i = p$ contains Z_j, so $Z^* = Z_j$.

Finally, since $(A_S)^H = \oplus_1^h (A_{N_j})^H$, it follows from what has just been shown that $\mathrm{Ker}(R^H \to (A_S)^H) = \bigcap_1^g z_i$. Q.E.D.

(4.7) COROLLARY. *With the notation of* (4.1), *the following statements hold*:

(4.7.1) *If for some* $j = 1,\dots, h$, $\{N' \cap R';\ N' \in \mathscr{S}'$ *and* $N' \cap A = N_j\} = \mathscr{M}'$, *then* $(A_{N_j})^H$ *contains and is integral over* R^H.

(4.7.2) *If* $\mathscr{M} = \mathscr{M}'$, *then* $(A_S)^H$ *contains and is integral over* R^H.

Proof. (4.7.1) By (4.6) and [6, (43.20)] $Z_j = (0)$, so $R^H \subseteq (A_{N_j})^H$. And the proof of (4.6) showed that $(A_{N_j})^H$ is integral over R^H/Z_j.

The proof of (4.7.2) is similar. Q.E.D.

(4.8) is the last of the preliminary results that are needed to prove the first of the main results in this section. It should be noted that the ring A of (4.1) plays no role in (4.8)—it is only necessary that a proper subset of $\mathscr{M}'$ be given. (In the proof we use (4.11.6), which could be proved here, but it is more closely related to the other results in (4.11).)

(4.8) PROPOSITION. *With the notation of* (4.1), *assume that* $\mathscr{M}' = \{M'_1,\dots, M'_k\}$ *with* $k > g$ *and that there exists an element* $x \in R'$ *such that* x *is quadratic over* R *and such that* x *is in* M'_i *if and only if* $i \leqslant g$. *Let* $P = (M, x) R[x]$ *and* Q *be the maximal ideals in* $R[x]$ *and let* $D = R[x]_P$ *and*

$L = R[x]_Q$. For $j = 1,\dots, k$ let z_j be the minimal prime ideal in R^H such that $(R^H/z_j)' = (R'_{M'_j})^H$. Then $D^H = R^H/(\bigcap_1^g z_i)$ and $L^H = R^H/(\bigcap_{g+1}^k z_i)$.

Proof. Since x is quadratic over R and since $\mathscr{M} \subset \mathscr{M}'$, it follows that $R[x]$ has exactly two maximal ideals, say P and Q. By the other property of x, one of them, say P, is $M'_i \cap R[x] = (M, x) R[x]$ for $i = 1,\dots, g$. Let $K_0 = \operatorname{Ker}(R^H \to D^H)$ and let $K = \operatorname{Ker}(R^H \to L^H)$, so $K_0 = \bigcap_1^g z_i$ and $K = \bigcap_{g+1}^k z_i$, by (4.6) applied to $R[x]$ in place of A. Therefore $R^H[x] = R[x]^H = D^H \oplus L^H \supseteq (R^H/K_0) \oplus (R^H/K) \supset R^H$, since R^H has only one maximal ideal and $(0) = K_0 \cap K$ in R^H, and so $R^H/K_0 = D^H$ and $R^H/K = L^H$, by (4.11.6). Q.E.D.

(4.9) is the first of the main results in this section. In regard to the hypothesis $\mathscr{M} \subset \mathscr{M}'$, note that if $\mathscr{M} = \mathscr{M}'$, then (3.3) (or its immediate generalization to the case when N is replaced by finitely many maximal ideals in A) is applicable.

(4.9) THEOREM. *With the notation of* (4.1), *assume that $\mathscr{M} \subset \mathscr{M}'$ and that there exists an element $x \in R'$ that is quadratic over R and is in a maximal ideal M' in R' if and only if $M' \in \mathscr{M}$. Let $P = M'_i \cap R[x]$ $(i = 1,\dots, g)$, so $P = (M, x)$ $R[x]$, and let $D = R[x]_P$. Then the following statements hold*:

(4.9.1) $D^H \subseteq (A_S)^H$.

(4.9.2) $D \subseteq A_S \cap F \subseteq (A_S)' \cap F = D'$.

(4.9.3) Altitude $A_S \cap F =$ altitude A_S.

(4.9.4) *A_S is not a flat R-module, so A is not a flat R-module.*

Proof. Note that (4.9.2)–(4.9.4) follow immediately from (4.9.1). For, if (4.9.1) holds and E is the quotient field of A, then $A_S \cap F = ((A_S)^H \cap E) \cap F = (A_S)^H \cap F \supseteq D^H \cap F = D$, and $A_S \cap F \subseteq (A_S)' \cap F = R'_{(R' - \cup\{M'_i\})} = D'$ by (4.2) and the hypothesis on $\mathscr{S}$ and x, so (4.9.2) holds. Then altitude $A_S \cap F =$ altitude $(A_S)' \cap F$, by (4.9.2) and integral dependence, and altitude$(A_S)' \cap F =$ altitude A_S, by (4.2), and so (4.9.3) holds. Moreover, (4.9.4) holds, since $R \subset A_S \cap F$ implies A_S is not R-flat, and so A is not R-flat. Therefore it remains to prove (4.9.1). However, $K = \operatorname{Ker}(R^H \to (A_S)^H) = \bigcap_1^g z_i$, by (4.6), and $R^H/K = D^H$, by (4.8), so $D^H \subseteq (A_S)^H$. Q.E.D.

Because of the quadratic hypothesis on x in (4.9), we consider the existence of such an element in the next few results. (In (4.10), QSM stands for quadratic separator of maximal ideals.) Actually, a (formally) weaker condition than that in (4.10) also implies the conclusions of (4.9) hold; concerning this, see (4.19). Some reasons for considering this stronger condition first are given in (4.11) (especially (4.11.3) and (4.11.4)) and in (4.15).

(4.10) DEFINITION. It will be said that a quasi-local ring R satisfies *condition* (QSM) in case there are only finitely many maximal ideals in R' and given any nonempty collection $\mathscr{M} = \{M'_1, \ldots, M'_g\}$ of maximal ideals in R', there exists an element $x \in R'$ that is quadratic over R such that $x \in \bigcap_1^g M'_i$ and $1 - x$ is in the other maximal ideals in R'.

It would be of interest (and, because of (4.12), of some importance) to know if every local domain satisfies condition (QSM). (4.11.3) and (4.11.4) contain some information in regard to this, and the other results in (4.11) are useful when considering condition (QSM).

(4.11) *Remarks.* (4.11.1) With the notation of (4.10), there exists a quadratic x as in (4.10) if and only if there exists a quadratic y in R' such that y is in a maximal ideal M' in R' if and only if $M' \in \mathscr{M}$.

(4.11.2) If R is a quasi-local ring such that R' has only one maximal ideal, then R satisfies condition (QSM).

(4.11.3) If R is a local domain, then there exists a finite integral extension domain $A \subseteq R'$ of R such that A_P satisfies condition (QSM) for all $P \in \operatorname{Spec} A$.

(4.11.4) If R is any quasi-local ring such that R' has only finitely many maximal ideals, then there exists a quasi-local integral extension ring R^* of R such that R^* is contained in a finite R-algebra $S \subseteq R'$ and such that R^* satisfies condition (QSM). (Thus, if R is Noetherian, then R^* is a finite R-algebra.)

(4.11.5) If R is a quasi-local domain such that R' has only finitely many maximal ideals, then the Henselization R^H of R satisfies condition (QSM). The completion R^* of R satisfies condition (QSM), if R^* has no imbedded prime divisors of zero.

(4.11.6) If (R, M) is a quasi-local ring and x is integral and quadratic over R, then $R[x]$ has at most two maximal ideals. If $R[x]$ has two maximal ideals and if S is a ring such that $R \subseteq S \subseteq R[x]$ and S has two maximal ideals, then $S = R[x]$.

Proof. The proof of (4.11.1) is straightforward, (4.11.2) is clear, and (4.11.3) follows immediately from (4.11.2) and [3, (23.2.5)].

(4.11.4) If R' has exactly h maximal ideals, then there exist $x_1, \ldots, x_h$ in R' such that $S = R[x_1, \ldots, x_h]$ has h maximal ideals. Therefore, with J the Jacobson radical of S, $R^* = R + J$ is quasi-local, and it is readily seen that R^* satisfies condition (QSM), since $J \subseteq R^*$.

(4.11.5) The zero ideal in R^H is a finite intersection of (minimal) prime ideals, by [6, (43.20)]. Let $M_1^*,\ldots, M_g^*$ be given maximal ideals in $R^{H'}$, and by (4.4.1) let z_i' be the minimal prime ideal contained in M_i^* $(i = 1,\ldots, g)$. Then $(0) = \bigcap_1^k z_i'$ in $R^{H'}$, where $R^{H'}$ has k maximal ideals. Then there exists an idempotent element $e \in \bigcap_1^g z_i'$ such that $1 - e \in \bigcap_{g+1}^k z_i'$, so condition (QSM) holds. A similar proof also works if R^* has no imbedded prime divisors of zero, since then its total quotient ring is Artinian.

(4.11.6) $R[x]$ has at most two maximal ideals, since $R[x]/MR[x]$ is quadratic over R/M. Assume $R[x]$ has two maximal ideals, say P' and Q', and let P and Q be the maximal ideals in S. Let $s \in P - Q$. Then $s = ax + b$ for some $a, b \in R$ and $R[ax] = R[s] \subseteq S$. If $a \in M$, then $ax \in P' \cap Q' \cap S = P \cap Q$, so $R[s] = R[ax]$ has only one maximal ideal (namely, $(M, ax) R[ax]$), in contradiction to the choice of s. Therefore, a is a unit in R, so $R[s] = R[ax] = R[x]$. Q.E.D.

(4.12) is essentially a restatement of (4.9) for the case when R satisfies condition (QSM).

(4.12) COROLLARY. *Let R be a quasi-local domain that satisfies condition* (QSM), *let A be an integral extension domain of R that has only finitely many maximal ideals, let $N_1,\ldots, N_h$ be maximal ideals in A, and let $S = A - \bigcup_1^h N_j$. Then the following statements hold*:

(4.12.1) *There exists x in R' that is quadratic over R and a maximal ideal P in $R[x]$ such that $D^H \subseteq (A_S)^H$, where $D = R[x]_P$.*

(4.12.2) *$D \subseteq A_S \cap F \subseteq (A_S)' \cap F = D'$, where D is as in* (4.12.1) *and F is the quotient field of R.*

(4.12.3) Altitude $A_S \cap F =$ altitude A_S.

(4.12.4) *If $A_S \cap F \nsubseteq R'$, then A_S and A are not R-flat.*

Proof. Let $\mathscr{S}'$ be the set of maximal ideals N' in A' such that $N' \cap A \in \{N_1,\ldots, N_h\}$ and let $\mathscr{M} = \{N' \cap R';\ N' \in \mathscr{S}'\}$. If every maximal ideal in R' is in $\mathscr{M}$, then let $x = 1$ and $D = R$, and the conclusions follow from (3.3) and (4.7). If some maximal ideal in R' is not in $\mathscr{M}$, then by condition (QSM) choose x for this set $\mathscr{M}$ that is quadratic over R. Then the conclusions follow from (4.9). Q.E.D.

(4.13) gives an interesting necessary condition for A to be R-flat when R satisfies condition (QSM).

(4.13) COROLLARY. *Let R be a quasi-local domain that satisfies condition* (QSM) *and let A be a flat integral extension domain of R that has only finitely many maximal ideals. Then for each pair of maximal ideals N in A and M' in R', there exists a maximal ideal N' in A' such that $N' \cap A = N$*

and $N' \cap R' = M'$. Therefore $(A_N)' \cap F = R'$, where F is the quotient field of R.

Proof. This follows immediately from (4.12.4) and (4.12.2) (and the proof of (4.12)) with $S = A - N$. Q.E.D.

(4.14) *Remark.* If R is any quasi-local domain such that R' has only finitely many maximal ideals and if A is a free quadratic integral extension domain of R, then the hypotheses of (4.13) are satisfied by $R^* = R + J$ (with J the Jacobson radical of R') and $A^* = R^*[A]$ in place of R and A, since it is clear that R^* satisfies condition (QSM) and that A^* is a free quadratic integral extension domain of R^*.

(4.18.3) contains another interesting fact about quasi-local domains that satisfy condition (QSM).

As already noted, I do not know if all local domains satisfy condition (QSM). However, when there exist height one maximal ideals in R', then there do exist elements in R' that are quadratic separators of a given collection of height one maximal ideals in R', as in shown in (4.15).

(4.15) THEOREM. *Let (R, M) be a local domain and assume there exists a height one maximal ideal in R'. Then for each collection $M'_1,\dots, M'_g$ of height one maximal ideals in R' there exists $x \in R'$ such that x is quadratic over R, $x \in \bigcap_1^g M'_i$, and $1 - x$ is in all other maximal ideals in R'.*

Proof. Let $M'_1,\dots, M'_k$ be all the maximal ideals in R' and let $z_1,\dots, z_k$ be the prime divisors of zero in R^H numbered so that z_i corresponds to M'_i $(i = 1,\dots, k)$, so $(R^H/z_i)' = (R'_{M'_i})^H$. Then $\bigcap_1^k z_i = (0)$ and depth $z_i = 1$ for $i = 1,\dots, g$. It may clearly be assumed $g < k$, so let $Z_1 = \bigcap_1^g z_i$, $Z_2 = \bigcap_{g+1}^k z_i$, and $Z = (Z_1, Z_2)R^H$. Then depth $Z_1 = 1$, so Z is primary for the maximal ideal M^H in R^H, and so $Z \cap R$ is M-primary. Therefore let $0 \neq b \in Z \cap R$, so $b = w + y$ for some $w \in Z_1$ and $y \in Z_2$, and $w, y \notin bR^H$, since R^H is flat over R and b is regular in R.

If altitude $R = 1$, then $R^H/bR^H = R/bR$, so there exists $c \in R$ such that $c - y \in bR^H$, hence $(b, c)R^H = (b, y)R^H$. This also holds if altitude $R > 1$, as will now be shown. Namely, since height $M'_1 = 1$, M'_1 is a prime divisor of bR', so $M = M'_1 \cap R$ is a prime divisor of bR, by [8, Theorem 2.15]. Therefore there exist ideals I and Q in R such that $bR = I \cap Q$, where $I: M = I$ and Q is M-primary. Also, $y \in \operatorname{Ker}(R^H \to R^H_P)$ for all $P \in \operatorname{Spec} R^H - \{z_1,\dots, z_g, M^H\}$, so y is in every primary component of IR^H (since $I \neq (0)$ and $I: M = I$), so $y \in IR^H$. Therefore $y \notin QR^H$ (since $y \notin bR^H$). Now $R^H/QR^H = R/Q$ and $(IR^H + QR^H)/QR^H = (I + Q)/Q$, so there exists $c \in I$ such that $c - y \in QR^H$. Therefore $c - y \in QR^H \cap IR^H = bR^H$, so $(b, c)R^H = (b, y)R^H$.

Now $b - y = w \in Z_1$ and $yw = 0$, so $y^2 = by$. Therefore, since $c - y \in bR^H$, it follows that $c^2 \in b(b,c)R^H \cap R = b(b,c)R$. Thus, if $x_0 = c/b$, then $x_0 \in R'$ and x_0 is quadratic over R. Also, $R[x_0]^H = R^H[x_0] = R^H[y/b]$ and, with T_0 the total quotient ring of R^H, $y/b \in z_i T_0 \cap R^H[x_0]$ for $i = g+1,\ldots,k$, and $w/b = 1 - (y/b) \in z_j T_0 \cap R^H[x_0]$ for $j = 1,\ldots,g$. Moreover, by the relationship between the z_i and the M_i', we have $z_i T_0 \cap R^{H'}$ is the unique minimal prime ideal contained in $M_i' R^{H'}$, since $(R'^H)_{M_i' R'^H} = (R'_{M_i'})^H = (R^H/z_i)' = R^{H'}/(z_i T_0 \cap R^{H'})$. Therefore it follows that $R[x_0]$ has exactly two maximal ideals and $M_1',\ldots,M_g'$ are the maximal ideals in R' that lie over one of them. Thus since each element in $R[x_0]$ is quadratic over R, the conclusion follows from (4.11.1). Q.E.D.

By combining (4.15) and (4.9), we have the following result.

(4.16) COROLLARY. *Let (R, M) be a local domain, let F be the quotient field of R, and let A be an integral extension domain of R that has only finitely many maximal ideals. Assume $\mathscr{S} = \{N_1,\ldots,N_h\}$ is a (finite) collection of height one maximal ideals in A and let $S = A - \bigcup_1^h N_i$. Then there exists $x \in R'$ that is quadratic over R and a maximal ideal P in $R[x]$ such that $D = R[x]_P \subseteq A_S \cap F \subseteq (A_S)' \cap F = D'$, so* altitude $A_S \cap F =$ altitude $A_S = 1$. *Moreover, if* altitude $R > 1$, *then A_S and A are not R-flat.*

Proof. Let $\mathscr{S}' = \{N';\ N'$ is a maximal ideal in A' and $N' \cap A \in \mathscr{S}\}$, let $\mathscr{M} = \{N' \cap R';\ N' \in \mathscr{S}'\}$, and let $\mathscr{M}'$ be the set of all maximal ideals in R'. Then all the ideals in $\mathscr{S}'$ and in $\mathscr{M}$ have height one, by integral dependence and [6, (10.14)]. If $\mathscr{M} = \mathscr{M}'$, then let $x = 1$, so $D = R \subseteq A_S \cap F \subseteq (A_S)' \cap F = \bigcap \{R'_{M'};\ M' \in \mathscr{M}'\} = R'$, by (4.2), and all these rings have altitude one by (4.2) and integral dependence. If $\mathscr{M} \subset \mathscr{M}'$, then let x be as in (4.15) for $\mathscr{M}$ and let $P = M' \cap R[x]$, where $M' \in \mathscr{M}$. Then the conclusions follow immediately from (4.9). Q.E.D.

(4.17) COROLLARY. *Let B be a Noetherian domain, let F be the quotient field of B, and let A be an integral extension domain of B. Then for all height one prime ideals P in A such that only finitely many $Q \in$* Spec *A lie over $P \cap B$,* altitude $A_P \cap F = 1$.

Proof. Let $p = P \cap B$. Then $PA_{(B-p)}$ is a height one maximal ideal and $R = B_p$ is a local domain, so the conclusion follows immediately from (4.16). Q.E.D.

If there exist elements in R' that are quadratic over R, then they give rise to an interesting relationship between certain factor rings of R^H and certain localities over R, as is shown in (4.18).

(4.18) *Remark.* Let (R, M) be a quasi-local domain such that R' has only finitely many maximal ideals, say $M_1',\ldots,M_k'$, let z_i be the minimal prime ideal in R^H such that $(R^H/z_i)' = (R'_{M_i'})^H$, $i = 1,..., k$, and let F be the quotient field of R. Then the following statements hold:

(4.18.1) If there exists $x \in R'$ that is quadratic over R and such that $R[x]$ has two maximal ideals, say P and Q, then let M_i' lie over P if and only if $i = 1,..., g < k$. Then, with $Q = R^H/(\bigcap_1^g z_i)$, $(Q \cap F)^H = Q$ and $Q \cap F = R[x]_P$.

(4.18.2) If R is Noetherian and depth $z_1 = \cdots =$ depth $z_g = 1$, then there exists such an x as in (4.18.1).

(4.18.3) If R satisfies condition (QSM), then for any nonempty proper subset $\{z_1,..., z_g\}$ of $\{z_1,..., z_k\}$ there exists such an x as in (4.18.1).

(4.18.4) If $(A_N)' \cap F = (A_N \cap F)'$ for all integral extension domains A of R and for all maximal ideals N in A, then $((R^H/z_i) \cap F)^H = R^H/z_i$ for each $i = 1,..., k$.

Proof. (4.18.1) follows immediately from (4.8) (and the fact that $D^H \cap F = D$).

(4.18.2) and (4.18.3) follows from (4.15) and (4.10), respectively.

For (4.18.4), fix i, let $L = R^H/z_i$, and let $D = L \cap F$. Then $R \subseteq D \subseteq L' \cap F = D'$, by hypothesis (since R^H is a localization of an integral extension ring of R), so D' is quasi-local, since L is Henselian. Since L dominates D, [6, (43.5)] implies that L dominates D^H/K for some (necessarily prime) ideal K in D^H and also that L' dominates $D'^H = D^{H\prime}$, so it follows that $K = (0)$. Also, D^H dominates R, so [6, (43.5)] implies D^H dominates R^H/K_0 for some prime ideal K_0 in R^H, so $R^H/z_i = L \supseteq D^H \supseteq R^H/K_0$. Therefore $K_0 = z_i$, and so $L = D^H$. Q.E.D.

The next result shows that a somewhat different condition than (QSM) still implies the conclusions of (4.9) hold. Note that by (4.18.1), this new condition is (formally) weaker than (QSM); I do not know if they are equivalent.

(4.19) PROPOSITION. *With the notation of* (4.1), *let* $\mathcal{M}' = \{M_1',..., M_k'\}$, *let* z_i *be the minimal prime ideal in* R^H *such that* $(R^H/z_i)' = (R'_{M_i'})^H$ $(i = 1,..., k)$, *and assume that R satisfies the following condition* (*): *for each subset* Z *of* $\{z_1,..., z_k\}$ *it holds that* $(L \cap F)^H = L$, *where* $L = R^H/(\bigcap \{z_i; z_i \in Z\})$. *Then the following statements hold*:

(4.19.1) $R^H/(\bigcap_1^g z_i) \subseteq (A_S)^H$.

(4.19.2) $C = (R^H/(\bigcap_1^g z_i)) \cap F \subseteq A_S \cap F \subseteq (A_S)' \cap F = C' = R'_{(R' - \bigcup_1^g M_i')}$.

(4.19.3) Altitude $A_S \cap F =$ altitude A_S.

(4.19.4) *If* $g < k$, *then* A_S *and* A *are not R-flat.*

Proof. This was shown to hold in the proof of (4.9)—note that it follows from condition (*) and (4.4.1) and (4.4.2) that $C' = R'_{(R' - \bigcup_i^g M_i')}$. Q.E.D.

(4.20) *Remark.* Condition (*) in (4.19) is equivalent to $(L \cap F)' = \bigcap \{R'_{M'}$; M' is a maximal ideal in R' and its corresponding minimal prime ideal in R^H is in $Z\}$.

Proof. Assume condition (*) holds, let Z be a given subset of $\{z_1,..., z_k\}$, let $L = R^H/(\bigcap \{z_i;\ z_i \in Z\})$, and let $C = L \cap F$. Then $C^H = L$, by hypothesis, so it follows from (4.4.1) and (4.4.2) that C' is as described.

Conversely, let Z, L, and C be as above and assume $C' = \bigcap \{R'_{M'}$; M' is a maximal ideal in R' and its corresponding minimal prime ideal in R^H is in $Z\}$. Then $C^{H'} = C'^H = L'$, by (4.4.1) and (4.4.2), and L dominates C^H/K for some (necessarily semi-prime) ideal K in C^H, by [6, (43.5)]. Therefore it follows that $K = (0)$. Also, C^H dominates R^H/K_0 for some semi-prime ideal K_0 in R^H, by [6, (43.5)], so $R^H/(\bigcap \{z_i;\ z_i \in Z\}) = L \supseteq C^H \supseteq R^H/K_0$. Therefore it follows that $K_0 = \bigcap \{z_i;\ z_i \in Z\}$, so $C^H = L$, and so condition (*) holds. Q.E.D.

(4.21) will be used to derive another corollary of (4.9) and also to relate the conditions in (4.23) to condition (*).

(4.21) LEMMA. *Let R be a quasi-local domain with quotient field F and assume the following condition holds: for all finite integral extension domains A of R and for all finite sets of maximal ideals $N_1,..., N_h$ in A, $(A_S)' \cap F = (A_S \cap F)'$, where $S = A - (N_1 \cup \cdots \cup N_h)$. Then this continues to hold for all integral extension domains B of R and for all finite sets of maximal ideals $Q_1,..., Q_h$ in B.*

Proof. Suppose there exists an integral extension domain B of R and finitely many maximal ideals $Q_1,..., Q_h$ in B such that $(B_S)' \cap F \supset (B_S \cap F)'$, where $S = B - (Q_1 \cup \cdots \cup Q_h)$. Let $L^* = (B_S)' \cap F$, let $D = B_S \cap F$, and let $x \in L^* - D'$. Then $x = a/bs$ with $a, b, s \in B$, $s \in S$, and $a/b \in B'$. Let $A = R[a, b, s]$, so $a/b \in A'$ and $s \notin U = \bigcup_1^h (Q_i \cap A)$. Therefore, with $S_0 = A - U$, $x \in (A_{S_0})' \cap F = (A_{S_0} \cap F)' \subseteq (B_S \cap F)' = D'$, and this is a contradiction, so the conclusion holds. Q.E.D.

(4.22) COROLLARY. *Let R be a quasi-local domain that satisfies condition (*), let F be the quotient field of R, let A be an integral extension domain of R, let $N_1,..., N_h$ be maximal ideals in A, and let $S = A - \bigcup_1^h N_i$. Then $(A_S)' \cap F = (A_S \cap F)'$.*

Proof. This is clear by (4.19.2) and (4.21). Q.E.D.

(4.23) gives two additional conditions that are closely related to $(A_S)' \cap F = (A_S \cap F)'$.

(4.23) PROPOSITION. *Let (R, M) be a quasi-local domain, let F be the quotient field of R, and consider the following statements:*

(4.23.1) *If A is an integral extension domain of R and P is a maximal ideal in A, then $(A_P)' \cap F$ is integral over $A_P \cap F$.*

(4.23.2) *If B is an integral extension domain of R and Q is a maximal ideal in B such that $B_Q \cap F \subseteq R'$, then $B_Q[R'] \cap F = R'$.*

(4.23.3) *If C is an integral extension domain of R and N is a maximal ideal in C, then $C_N \cap F \subseteq R'$ only if the following condition holds for at least one collection $\mathscr{M}$ of maximal ideals M' in R' such that $R' = \bigcap \{R'_{M'};\ M' \in \mathscr{M}\}$: for each $M' \in \mathscr{M}$ there exists a maximal ideal N' in C' such that $N' \cap R' = M'$ and $N' \cap C = N$.*

Then (4.23.1) $\Rightarrow$ (4.23.2) $\Leftrightarrow$ (4.23.3). *Moreover, if* (4.23.3) *holds for all quasi-local extension domains $D \subseteq F$ of R in place of R, then all three statements hold for all such quasi-local domains D.*

Proof. Assume (4.23.1) holds and let B and Q be as in (4.23.2), so $B_Q \cap F \subseteq R'$. Then $B_Q \cap F \subseteq B_Q[R'] \cap F \subseteq (B_Q)' \cap F = (B_Q \cap F)' = R'$, by (4.23.1) and hypothesis, so $B_Q[R'] \cap F \subseteq R'$. Therefore (4.23.2) holds, since the other inclusion is clear.

Assume (4.23.2) holds and let C and N be as in (4.23.3), so $C_N \cap F \subseteq R'$. Now $C_N[R'] = C[R']_{(C-N)} = R'[C]_{(C-N)}$, so $(R'[C]_{(C-N)}) \cap F = C_N[R'] \cap F = R'$, by (4.23.2). Let $\mathscr{S}'$ be the set of maximal ideals N' in $R'[C]$ that lie over N. Then $R' = (R'[C]_{(C-N)}) \cap F = \bigcap \{R'[C]_{N'};\ N' \in \mathscr{S}'\} \cap F = \bigcap \{R'_{N' \cap R'};\ N' \in \mathscr{S}'\}$, by [2, (12.7)]. Therefore it follows that (4.23.3) holds with $\mathscr{M} = \{N' \cap R';\ N' \in \mathscr{S}'\}$.

Assume (4.23.3) holds and let B and Q be as in (4.23.2), so $B_Q \cap F \subseteq R'$. Let $\mathscr{S}'$ be the collection of all maximal ideals Q' in B' that lie over Q and let $\mathscr{M}$ be the collection of maximal ideals in R' given by (4.23.3) for B and Q. Then $(B_Q)' = \bigcap \{B'_{Q'};\ Q' \in \mathscr{S}'\}$, so $R' \subseteq (B_Q)' \cap F = \bigcap \{B'_{Q'};\ Q' \in \mathscr{S}'\} \cap F \subseteq \bigcap \{R'_{M'};\ M' \in \mathscr{M}\} = R'$, by (4.23.3), so it readily follows that $B_Q[R'] \cap F = R'$, and so (4.23.3) $\Rightarrow$ (4.23.2).

Finally, assume (4.23.3) holds for all quasi-local extension domains $D \subseteq F$ of R in place of R and let (D, N) be such a ring. Let A be an integral extension domain of D, let P be a maximal ideal in A, let $L = A_P \cap F$, and let $B = L[A]$. Then L is a quasi-local extension domain of R contained in F, B is integral over L, $p = PA_P \cap B$ is a maximal ideal in B, and $B_p \cap F = A_P \cap F = L \subseteq L'$. Therefore, by assumption, (4.23.3) holds for L, so there exists a collection $\mathscr{M}$ of maximal ideals Q in L' such that $L' = \bigcap \{L'_{Q'};\ Q' \in \mathscr{M}\}$ and such that for each maximal ideal $Q' \in \mathscr{M}$, there exists a maximal ideal p' in B' such that $p' \cap L' = Q'$ and $p' \cap B = p$. Let $\mathscr{S}'$ be the collection of all maximal ideals p' in B' that lie over p. Then $L' \subseteq (B_p)' \cap F = \bigcap \{B'_{p'};\ \ p' \in \mathscr{S}'\} \cap F \subseteq \bigcap \{L'_{Q'};\ \ Q' \in \mathscr{M}\} = L'$, so

$(B_p)' \cap F = L'$. But $B_p = A_P$ and $L = A_P \cap F$, so $(A_P)' \cap F = (A_P \cap F)'$, and so (4.23.1) holds for all such D. Therefore, by what has already been shown, (4.23.2) and (4.23.3) also hold for all such D. Q.E.D.

(4.24) *Remark.* (4.24.1) If, in (4.23), it is only assumed that (4.23.3) holds for all quasi-local extension domains $L \subseteq F$ of R of the form $A_P \cap F$, with A integral over R and P a maximal ideal in A, then a similar proof shows that (4.23.1) holds for R.

(4.24.2) It is clear by (4.23) that if any of the three statements in (4.23) holds for all quasi-local domains, then all three statements hold for all such rings.

(4.24.3) It should be noted that if there are only finitely many maximal ideals in R', then condition (*) in (4.19) implies (4.23.1), by (4.22), and (4.23.1) implies the weak version of condition (*) given in (4.18.4), by (4.18.4). I do not know if these three conditions are, in fact, equivalent.

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